Self Supervised Learning Methods for Imaging

Part 4: Learning with equivariance

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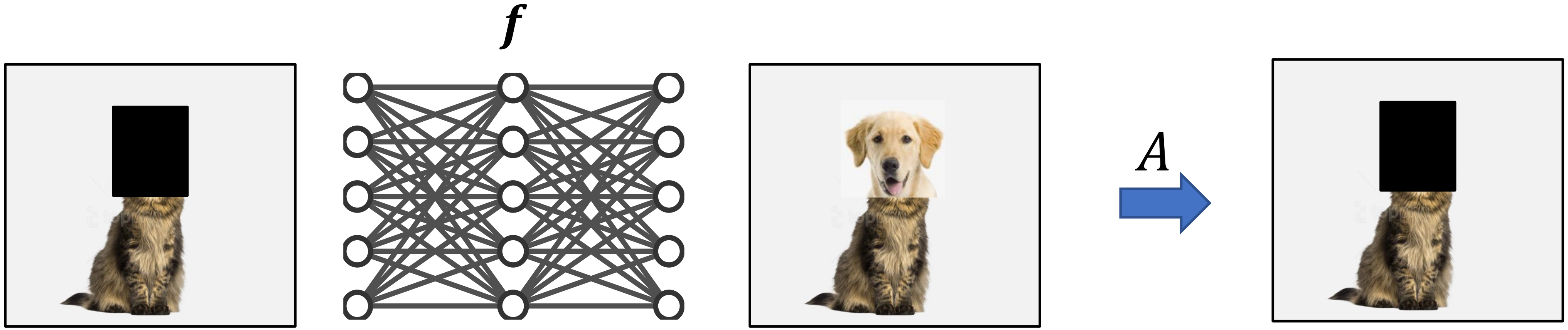
Learning Approach

Recall:

Proposition: Any reconstruction function $f(\boldsymbol{y}) = A^{\dagger}\boldsymbol{y} + g(\boldsymbol{y})$ where $g: \mathbb{R}^m \mapsto \mathcal{N}_A$ is any function whose image belongs to the nullspace of A is measurement consistent.



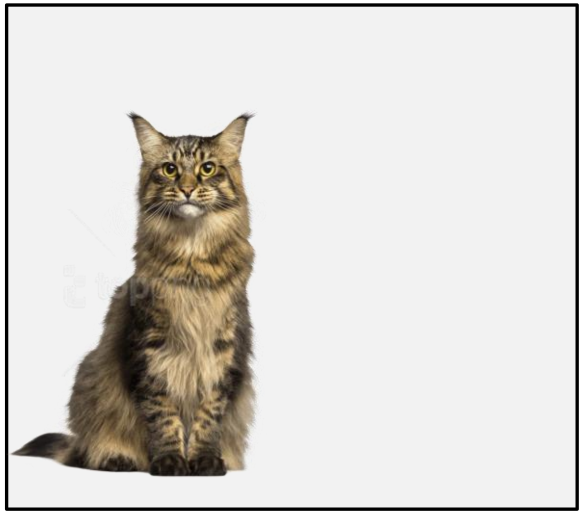
Symmetry Prior

Idea: Most natural signals sets $\mathcal{X}$ are invariant to groups of transformations.

Example: natural images are translation invariant

- Mathematically, a set $\mathcal{X}$ is invariant to $\{T_g \in \mathbb{R}^{n\times n}\}_{g\in G}$ if

$$\forall \boldsymbol{x} \in \mathcal{X},\ \ \forall g \in G,\ T_g\boldsymbol{x} \in \mathcal{X}$$



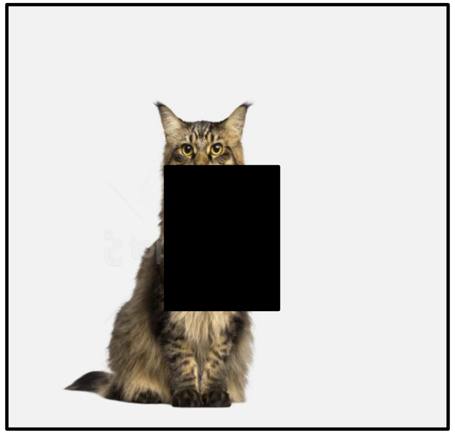
Other symmetries: rotations, permutation, amplitude

Symmetry Prior

Equivariant Imaging [Chen et al., 2021]

For all $g \in G$ we have

$$\boldsymbol{y} = A\boldsymbol{x} = \underbrace{AT_g}_{A_g}\overbrace{T_g^{-1}\boldsymbol{x}}^{x'} = A_g\boldsymbol{x}'$$



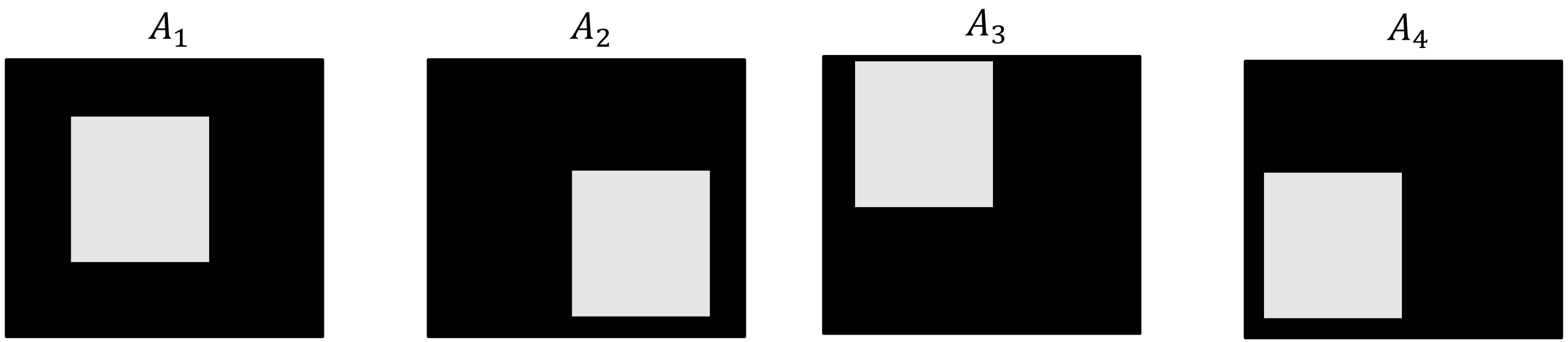
- We get multiple virtual operators $\{A_g\}_{g\in G}$ ‘for free’!
- Each AT_g might have a different nullspace

Necessary condition

Proposition [T. et al., 2023]: Learning reconstruction mapping f from observed measurements possible only if

$$\text{rank}\left(\mathbb{E}_g\, T_g^\top A^\top A T_g\right) = n,$$

and thus if $m \geq \max \frac{c_j}{s_j} \geq \frac{n}{|G|}$ where s_j and c_j are dimension and multiplicity of irreps.



(Non)-Equivariant Operators

Theorem [T. et al., 2023]: The full rank condition requires that A **is not equivariant:** $AT_g \neq \tilde{T}_g A$

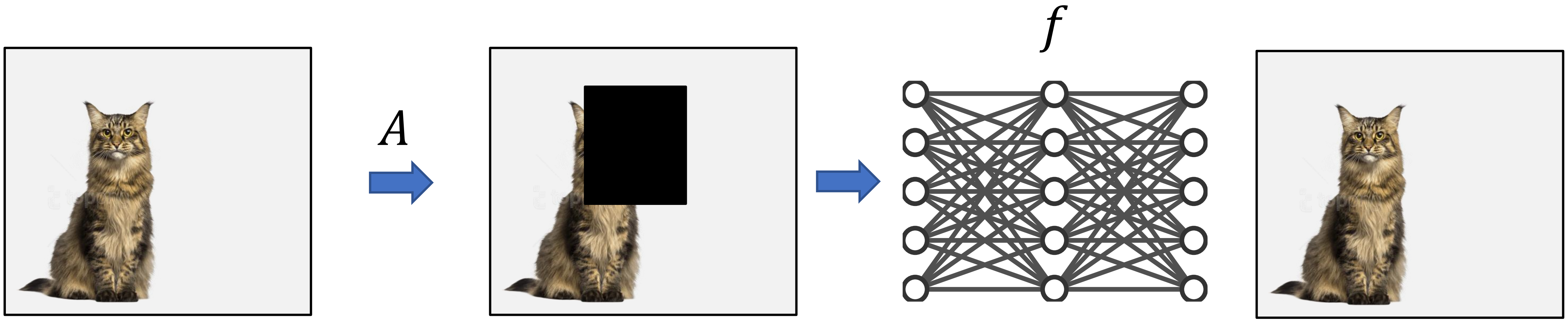
$$\operatorname{rank}\left(\mathbb{E}_g\, T_g^\top A^\top A T_g\right) = \operatorname{rank}\left(A^\top (\mathbb{E}_g \tilde{T}_g^\top \tilde{T}_g) A\right) = \operatorname{rank}\left(A^\top A\right) = m < n$$

	Translation	Rotation	Scaling	Amplitude
Gaussian Blur	★	★		★
Image Inpainting				★
Sparse-view CT	★		★	★
Accelerated MRI	★			★
Downsampling (with antialias)	★	★		★

Equivariant Imaging

How can we enforce equivariance in practice?

Idea: we should have $f\left(AT_g\boldsymbol{x}\right) = T_g f(A\boldsymbol{x})$, i.e. $f \circ A$ should be G-equivariant



Equivariant Imaging

How can we enforce equivariance in practice [Chen, 2021]?

$$\mathcal{L}_{EI}(\boldsymbol{y}, f) = \mathbb{E}_g \, || T_g\hat{\boldsymbol{x}} - f\left(AT_g\hat{\boldsymbol{x}}\right)||^2$$

where $\hat{\boldsymbol{x}} = f(\boldsymbol{y})$ is used as reference

Proposition [T. & Pereyra, *2024*]**:** *For linear and measurement consistent* $Af(A\boldsymbol{x}) = A\boldsymbol{x}$ *reconstruction, we have*

$$\mathcal{L}_{EI}(\boldsymbol{y}, f) = ||\boldsymbol{x} - f(\boldsymbol{y})||^2 + bias$$

where the *bias* term is small if $f \circ A$ is **not** equivariant.

Combining Losses

Robust Equivariant Imaging [Chen et al., *2022*]

enforces equivariance of $f \circ A$

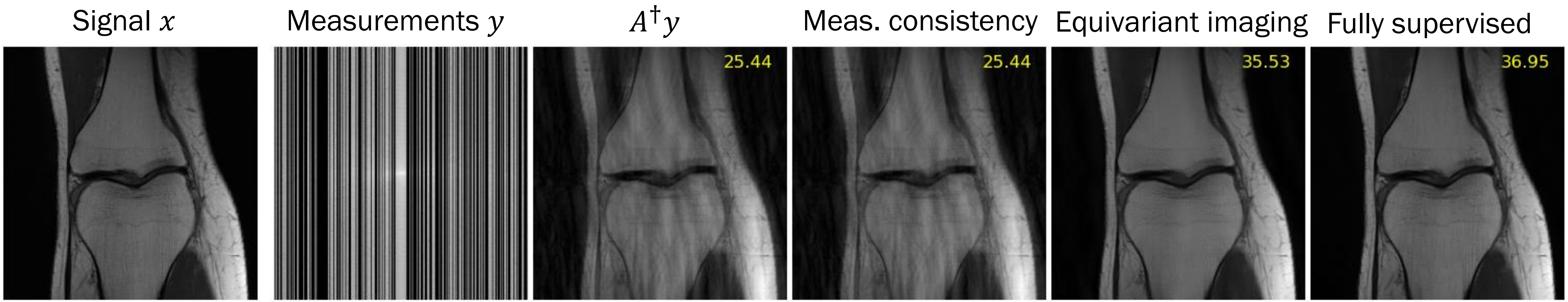
$$\mathcal{L}_{\text{REI}}(\boldsymbol{y}, f) = \underbrace{\mathcal{L}_{\text{SURE}}(\boldsymbol{y}, f)} + \overbrace{\mathcal{L}_{\text{EI}}(\boldsymbol{y}, f)}$$

unbiased estimator of 'noiseless' measurement consistency

- SURE can be replaced by any other noise-robust loss (eg. Noise2Void, etc.)

Magnetic Resonance Imaging

- Operator A is a subset of Fourier measurements (x2 downsampling)
- Dataset is approximately **rotation invariant**



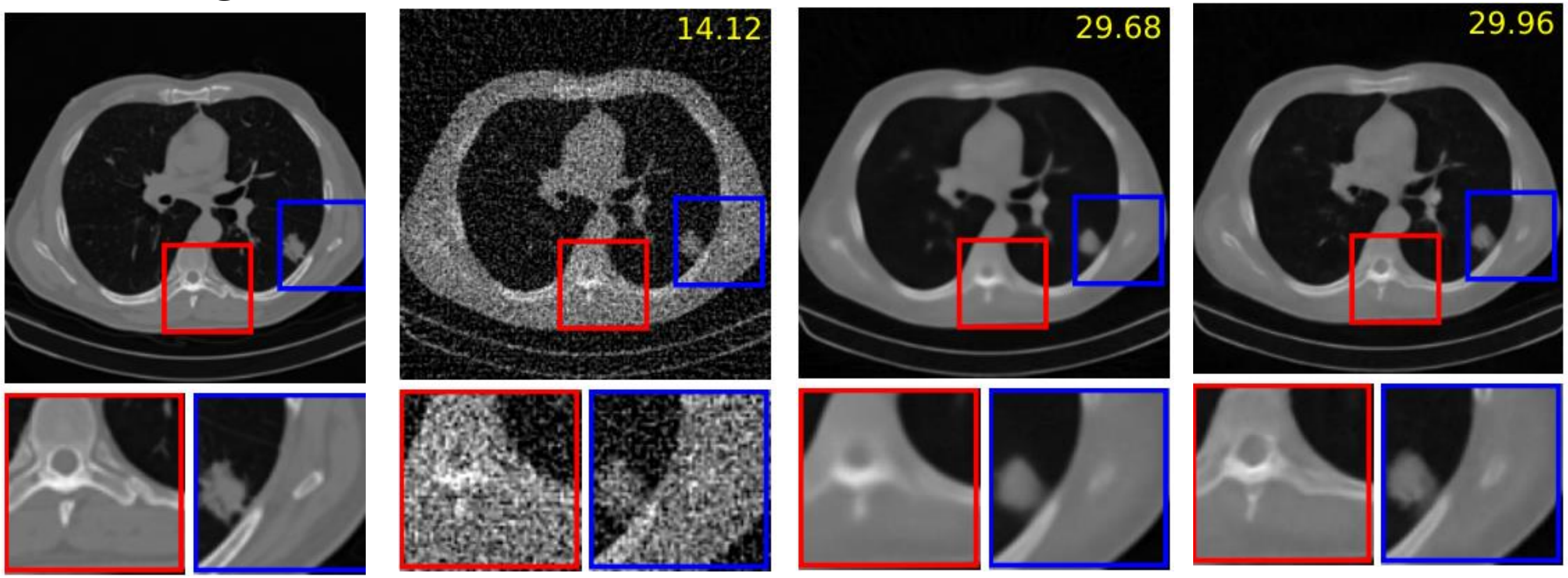
Chen, T., Davies, *CVPR* 2022

Computed Tomography

- Operator A is (non-linear variant) sparse radon transform
- Mixed Poisson-Gaussian noise
- Dataset is approximately **rotation invariant**

Noisy measurements y

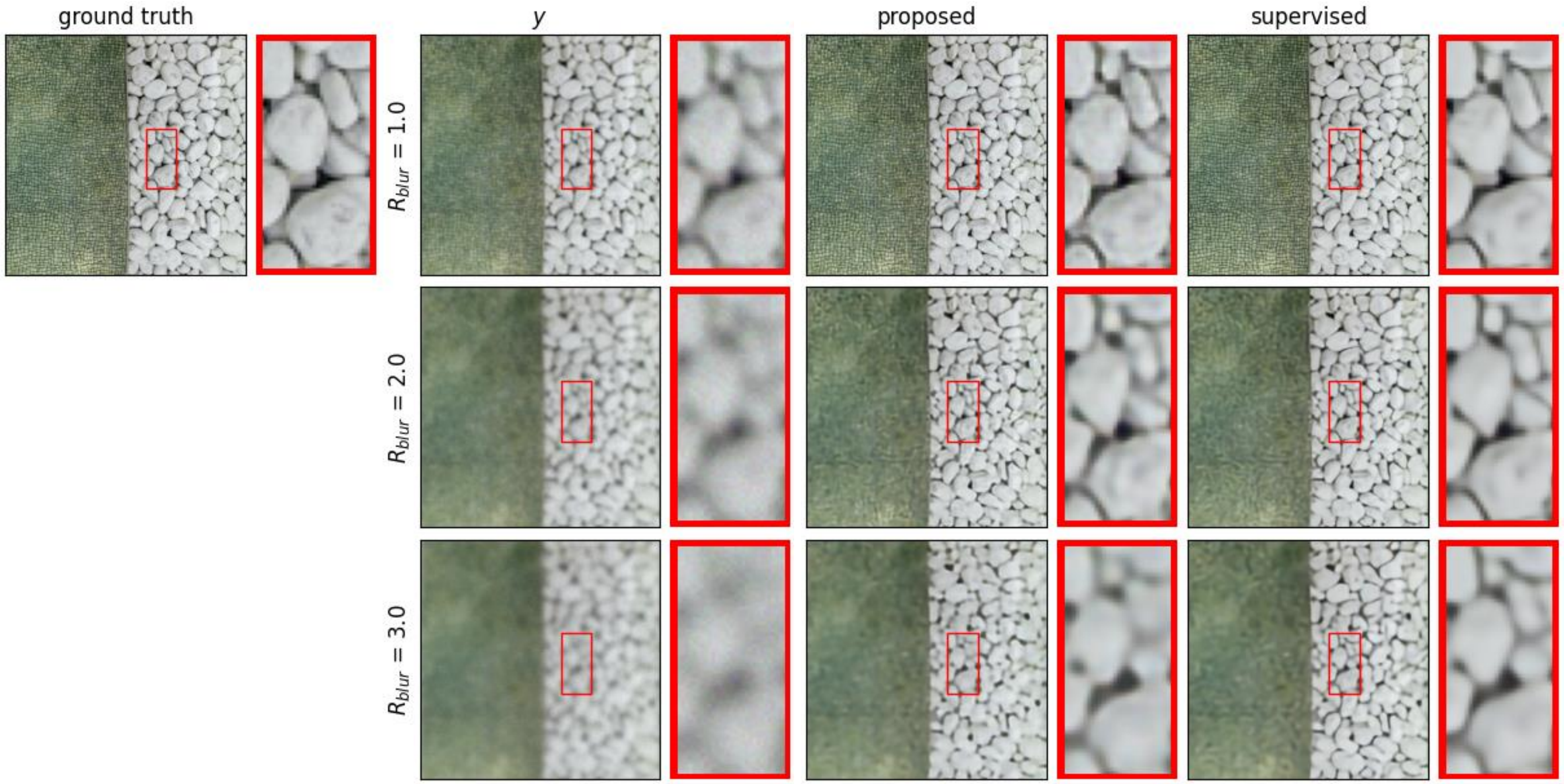
Clean signal x | Meas. consistency | Robust EI | Supervised



Chen, T., Davies, *CVPR* 2022

Image Deblurring

- Operator A is isotropic blur with Gaussian noise
- Dataset is approximately **scale invariant**



Scanvic, Davies, Abry, T., *arxiv* 2023

References

The full reference list for this tutorial can be found here:

https://tachella.github.io/projects/selfsuptutorial/