Unsupervised Learning From Incomplete Measurements for Inverse Problems

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NEURAL INFORMATION PROCESSING SYSTEMS

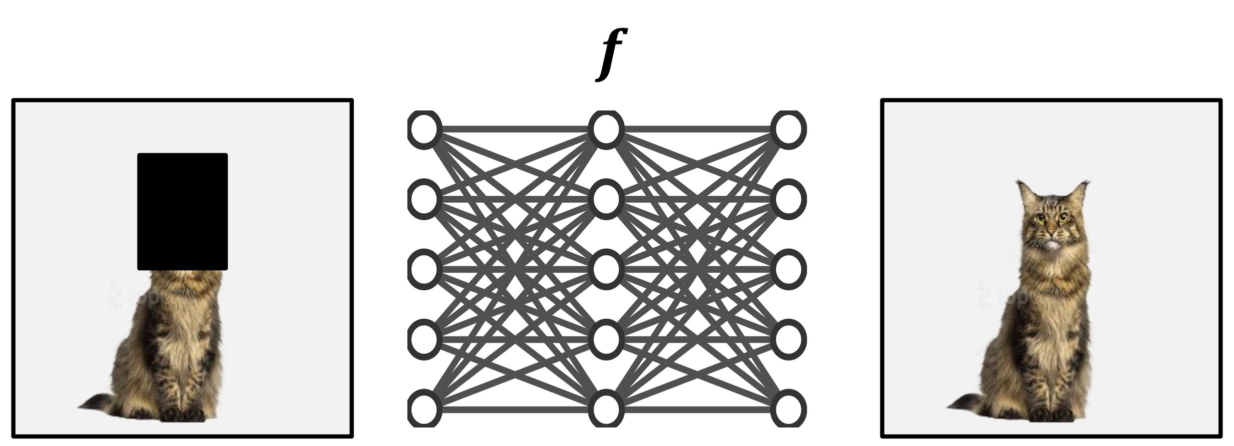
Problem setup

We consider incomplete linear observations of the form

$$y_i = A_{g_i} x_i$$

- Measurement $y_i \in \mathbb{R}^m$
- Signal $x_i \in \mathcal{X} \subset \mathbb{R}^n$
- Incomplete linear operators $A_{g_i} \in \mathbb{R}^{m\times n}$ with $g_i \in \{1,\dots,G\}$

Goal: Learn the reconstruction function $f: y \mapsto x$ from data

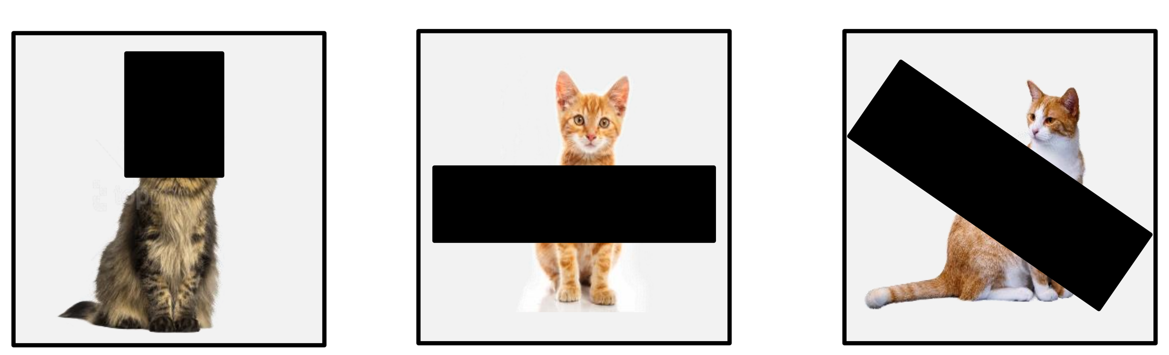


- Supervised dataset $\mathfrak{D}_s = \{(x_i, y_i, A_{g_i})\}_{i=1}^N$ *expensive or hard to obtain!*
- Unsupervised dataset: $\mathfrak{D}_u = \{(y_i, A_{g_i})\}_{i=1}^N$

Examples

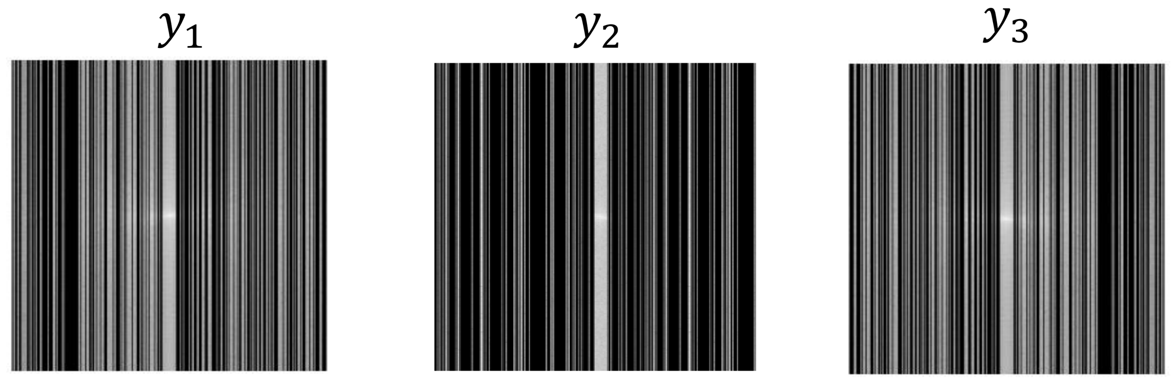
- Image inpainting

y_1 y_2 y_3



- Accelerated magnetic resonance imaging

y_1 y_2 y_3



Theory

Signal Recovery: is the reconstruction function $f: y \mapsto x$ one-to-one with known $\mathcal{X}$?

Theorem [1]: *A signal x belonging to a known set $\mathcal{X}$ with box-counting dimension $k < n$ can be uniquely recovered from the measurement $y = Ax$ with almost every $A \in \mathbb{R}^{m\times n}$ if*

$$m > 2k.$$

Most models are low-dimensional: Sparse dictionaries, manifold models, generative models, etc.

Model Identification: can we identify $\mathcal{X}$ from measurement data alone?

Necessary Condition

Proposition: *Identifying $\mathcal{X}$ from observed measurement sets $\{\mathcal{Y}_g = A_g\mathcal{X}\}_{g=1}^G$ with $A_1,\dots,A_G \in \mathbb{R}^{m\times n}$ possible only if*

$$\operatorname{rank}\left(\begin{bmatrix} A_1 \\ \vdots \\ A_G \end{bmatrix}\right) = n$$

and thus, if $m \geq n/G$.

Sufficient Condition

Theorem: *Identifying a set $\mathcal{X}$ with box-counting dimension $k < n$ from observed sets $\{\mathcal{Y}_g = A_g\mathcal{X}\}_{g=1}^G$ is possible by almost every $A_1,\dots,A_G \in \mathbb{R}^{m\times n}$ if*

$$m > k + n/G.$$

Proposed Self-Supervised Training Loss

$$\underset{f}{\operatorname{argmin}} \sum_i \left\| y_i - A_{g_i} f(y_i, A_{g_i}) \right\|^2 + \sum_s \left\| f(A_s\hat{x}_i, A_s) - \hat{x}_i \right\|^2 \text{ with } \hat{x}_i = f(y_i, A_{g_i}).$$

Enforces measurement consistency $y = A_g f(y, A_g)$

Enforces cross-operator consistency $f(A_g x, A_g) = f(A_s x, A_s)$ for all $s \neq g$

Experiments

MNIST dataset:

- Red line: sufficient condition $m > k + n/G$ with $k \approx 12$ [2]
- Test PSNR in dB, f is a fully-connected network

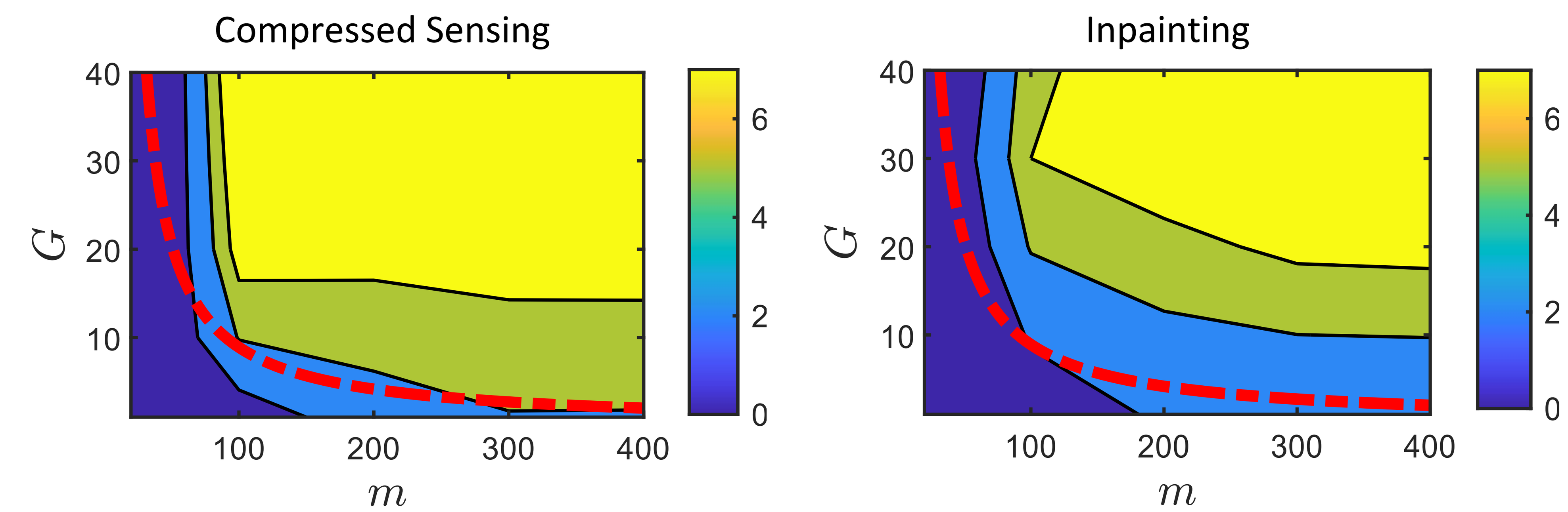


Image Inpainting with CelebA dataset

- $m = n/2$, $G = 40$, f is a U-Net

	$A^\dagger y$	AmbientGAN [3]	MOI (ours)	Supervised
Test PSNR	9.05 ± 1.65	29.57 ± 1.24	$\mathbf{34.05 \pm 3.77}$	36.21 ± 3.76

Accelerated MRI with FastMRI dataset

- $m = n/4$, $G = 40$, f is an unrolled prox. grad descent network

	$A^\dagger y$	Meas. Splitting [4]	MOI (ours)	Supervised
Test PSNR	25.77 ± 2.71	29.47 ± 2.02	$\mathbf{31.39 \pm 2.17}$	32.42 ± 2.44

References

[1] Sauer, Yorke and Casdagli (1991). "Embedology." In Journal of statistical Physics.

[2] Hein and Audibert (2005). "Intrinsic dimensionality estimation of submanifolds in $\mathbb{R}^d$." In International Conference on Machine learning.

[3] Bora, Price and Dimakis (2018). "AmbientGAN: Generative models from lossy measurements." In International Conference on Learning Representations.

[4] Yaman, Hosseini, Moeller, Ellermann, Ugurbil and Akçakaya. (2020) "Self-supervised learning of physics-guided reconstruction neural networks without fully sampled reference data." In Magnetic Resonance in Medicine.

Project webpage